Exit problem and its PDE characterization

Qingshuo Song (CityU HK)

Working paper with Yuecai Han (Jilin Univ) and Gu Wang (WPI)

@IMA workshop on Financial and Economic Applications, June 2018

Outline

Outline

Problem setup

Does a Feynman-Kac functional solve Dirichlet problem?

1. Feynman-Kac functional is

$$v(x) := \mathbb{E}\Big[\int_0^\zeta e^{-\lambda s} \ell(X(s)) ds + e^{-\lambda \zeta} g(X(\zeta)) \Big| X(0) = x \Big]$$

2. Associated Dirichlet PDE is

$$-\mathcal{L}u(x) + \lambda u(x) - \ell(x) = 0 \text{ on } O, \text{ with } u = g \text{ on } O^c.$$

(Q). Does v solve PDE?

- X is Cádlàg Feller with generator $\mathcal{L}$, i.e. $X \sim \mathcal{L}$;
  O is a connected bounded open set in $\mathbb{R}^d$;
  ζ is exit time from $\bar{O}$, denoted by $\zeta = \tau_{\bar{O}}(X)$.
- See for traditional time-dependent Feynman-Kac formula at
  B Oksendal. Stochastic differential equations. 2003

Example 1

Solvability question

Let $O = (0,1)$, $X \sim \mathcal{L}u := \frac{1}{2}\epsilon^2 u'' + u'$ and $\zeta = \tau_{\bar{O}}(X)$.

(Q) Does

$$v(x) = \mathbb{E}\Big[\int_0^\zeta e^{-s} 1 ds \Big| X(0) = x\Big],$$

solve ODE below?

$$-u' - \frac{1}{2}\epsilon^2 u'' + u - 1 = 0 \text{ on } O, \text{ and } u(0) = u(1) = 0.$$

Def A function $u \in C(\bar{O})$ is said to be a viscosity solution, if

1. u satisfies the viscosity solution property at each $x \in O$
2. $u = g$ at each $x \in \partial O$.

Example 1

Answer: Yes iff $\epsilon > 0$.

If $\epsilon = 0$, the explicit computation of

$$v = -e^{-1+x} + 1$$

does NOT satisfy the boundary condition $u(0) = 0$, while it satisfies $u(1) = 0$.

i.e. v loses its boundary at 0.

$X \sim \mathcal{L}u := \frac{1}{2}\epsilon^2 u'' + u'$
$v(x) = \mathbb{E}\Big[\int_0^\zeta e^{-s} 1 ds \Big| X(0) = x\Big]$,
$-u' - \frac{1}{2}\epsilon^2 u'' + u - 1 = 0$ on O, and $u(0) = u(1) = 0$.
If $\epsilon > 0$, then

$$v(x) = 1 + \frac{(1 - e^{\lambda_1})e^{\lambda_2 x} + (e^{\lambda_2} - 1)e^{\lambda_1 x}}{e^{\lambda_1} - e^{\lambda_2}},$$

where

$$\lambda_1 = \frac{\sqrt{1 + 2\epsilon^2} - 1}{\epsilon^2}, \text{ and } \lambda_2 = \frac{-\sqrt{1 + 2\epsilon^2} - 1}{\epsilon^2}.$$

Example 1

Literature review: Sufficient condition for solvability

[BS18] v solves PDE if all points on ∂O is regular to $\bar{O}^c$.

In Example 1,

$$O = (0,1), \ X \sim \mathcal{L}u = \frac{1}{2}\epsilon^2 u'' + u'.$$

- If $\epsilon > 0$, then both 0 and 1 are regular;
- If $\epsilon = 0$, then both 1 is regular, but 0 is irregular.
  Thus, $v(0)$ does not satisfy the boundary condition.
  But, v satisfies viscosity solution property at $x = 0$, which means ...

[BS18] E Bayraktar, Q Song, Solvability of Dirichlet Problems, SICON 2018.

x is *regular* to $\bar{O}^c$ w.r.t. $\mathcal{L}$, if $\mathbb{P}^x(\zeta = 0) = 1$.

Example 1

Definition of the viscosity property

In Exm 1 with $\epsilon = 0$,

- $x = 0$ is irregular and satisfy viscosity solution property.
- Therefore, v is a generalized viscosity solution in the sense of ...

Supertest functions: $J^+(u,x) = \{\phi \in C_0^\infty(\mathbb{R}^d), \text{ s.t. } \phi \geq (uI_{\bar{O}} + gI_{\bar{O}^c})^* \text{ and } \phi(x) = u(x)\}$.
Subtest functions: $J^-(u,x) = \{\phi \in C_0^\infty(\mathbb{R}^d), \text{ s.t. } \phi \leq (uI_{\bar{O}} + gI_{\bar{O}^c})_* \text{ and } \phi(x) = u(x)\}$.
With $G(\phi, x) = -\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x)$, consider

$$G(u,x) = 0, \text{ on } O \text{ and } u = 0 \text{ on } O^c.$$

1. $u \in USC(\bar{O})$ satisfies the viscosity subsolution property at some $x \in \bar{O}$, if
$$G(\phi, x) \leq 0, \ \forall \phi \in J^+(u,x).$$
2. $u \in LSC(\bar{O})$ satisfies the viscosity supersolution property at some $x \in \bar{O}$, if
$$G(\phi, x) \geq 0, \ \forall \phi \in J^-(u,x).$$

Main objective

Definition of the generalized viscosity solution

Def. A function $u \in C(\bar{O})$ is said to be a *generalized viscosity solution*, if

1. At each $x \in O$, u satisfies the viscosity solution property
2. At each $x \in \partial O$, u satisfies either the viscosity solution property or $u = g$

Goal Is v a generalized viscosity solution of PDE?

$$v(x) = \mathbb{E}\Big[\int_0^{\zeta} e^{-\lambda s}\ell(X(s))ds + e^{-\lambda s} g(X(\zeta)) \Big| X(0) = x \Big]$$

$$-\mathcal{L}u(x) + \lambda u(x) - \ell(x) = 0 \text{ on } O, \text{ with } u = g \text{ on } O^c.$$

Outline

A restatement of generalized viscosity solution

Boundary decomposition is a part of unknown

A function $u \in C(\bar{O})$ is said to be a generalized viscosity solution, if u satisfies the viscosity solution property at each $x \in \bar{O} \setminus \Gamma_{out}$, where

$$\Gamma_{out} = \{x \in \partial O : u = g\}.$$

First sufficient condition

If $v \in C(\bar{O})$, then ...

Lem If $v \in C(\bar{O})$, then v is a generalized viscosity solution of PDE with

$$\Gamma_{out} \supset \bar{O}^{c,*} \cap \partial O.$$

Rmk $\bar{O}^{c,*}$ is fine closure of $\bar{O}^c$, which means ...

Suppose $v \in C(\bar{O})$.

- If $x \in O$, then v satisfies viscosity solution property:
  $\mathbb{P}^x(\zeta > 0) = 1$ and Take Ito's formula on $\phi(X^x)$ for test functions ϕ.
- If $x \in \partial O$ and $\mathbb{P}^x(\zeta > 0) = 1$, then v satisfies viscosity solution property:
  Similar to above, take Ito's formula on $\phi(X^x)$ for test functions ϕ.
- If $x \in \partial O$ and $\mathbb{P}^x(\zeta > 0) = 0$, then $v(x) = g(x)$ by definitioin.
- $x \in \partial O$ and $\mathbb{P}^x(\zeta > 0) \in (0, 1)$ is not possible by Blumenthal 0-1 law.

Regularity and Fine topology

Facts on fine topologies refer to Section 3.4 of [CW05].

- A point x is regular (w.r.t. $\mathcal{L}$) for the set B if $\mathbb{P}^x(\tau_{B^c} = 0) = 1$;
- B^r denotes the set of all regular points for B;
- $B^* = B \cup B^r$ is called fine closure of B;
- A set A is finely open if $\mathbb{P}^x(\tau_A > 0) = 1$ for all $x \in A$.

[CW05] K Chung and J Walsh. Markov processes, Brownian motion, and time symmetry, Springer 2005

In Example 1,

$$O = (0, 1), \; X \sim \mathcal{L}u = \frac{1}{2}\epsilon^2 u'' + u'.$$

Set $B = \bar{O}^c$ and $B^c = \bar{O}$.
If $\epsilon > 0$, then $B^r = B^* = \bar{B}$ and $\Gamma_{out} \supset \{0, 1\}$;
If $\epsilon = 0$, then $B^r = B^* = B \cup \{1\}$ and $\Gamma_{out} \supset \{1\}$;

Find an example where B^r is a proper subset of B^*.

Example 2

$v \notin C(O) \subset C(\bar{O})$

- Let $O = (-1, 1) \times (0, 1)$, $X \sim \mathcal{L} = \partial_{x_1} + 2x_1 \partial_{x_2}$.
- Does
$$v(x) = \mathbb{E}\Big[\int_0^\zeta e^{-s} 1 ds \Big| X(0) = x\Big]$$
solve PDE in general viscosity sense?
$$-\partial_{x_1} u(x) - 2x_1 \partial_{x_2} u(x) + u(x) - 1 = 0, \text{ on } O, \text{ and } u(x) = 0 \text{ on } O^c.$$

(A) No.
(Q) When is $v \in C(\bar{O})$?

Particularly, $v(x) = 1 - e^{-\zeta^x}$ is discontinuous at every point on the curve $\partial O_1 \cap \partial O_3$, where

$$\zeta^x = -x_1 + \sqrt{1 - x_2 + x_1^2}, \ \forall x \in O_1 := \{x_2 \geq x_1^2\} \cap \bar{O},$$

$$\zeta^x = 1 - x_1, \ \forall x \in O_2 := \{x_2 < x_1^2, x_1 > 0\} \cap \bar{O},$$

$$\zeta^x = -x_1 - \sqrt{-x_2 + x_1^2}, \ \forall x \in O_3 := \{x_2 < x_1^2, x_1 < 0\} \cap \bar{O}.$$

Main result

Recall $\hat{\zeta} = \tau_O(X)$ and $\zeta = \tau_{\bar{O}}(X)$.

Thm If $\mathbb{P}^x(X(\hat{\zeta}) \in \bar{O}^{c,*}) = 1$ for all $x \in \bar{O}$, then v is a g.v.s. of PDE with

$$\Gamma_{out} \supset \bar{O}^{c,*} \cap \partial O.$$

In particular, if $\bar{O}^{c,*} = O^c$, then $\Gamma_{out} = \partial O$ and v is a v.s.

Why does Example 2 violate the condition?

Outline

Preliminaries on α-stable process

Suppose

- $J \sim -(-\Delta)^{\alpha/2}$ is an α-stable process for some $\alpha \in (0, 2)$;
- $dX_t = bdt + \sigma dJ_t$ is $X \sim \mathcal{L} := b \cdot \nabla - |\sigma|^\alpha (-\Delta)^{\alpha/2}$ for some $\sigma > 0$;
- O be a bounded open set satisfying exterior cone condition.

Then, all points in O^c is regular for $\bar{O}^c$ (i.e. $O^c = \bar{O}^{c,*}$) iff

$$\text{either } \alpha \geq 1 \text{ or } b = 0 \text{ holds.}$$

Linear non-stationary equation

Consider

$$\partial_t u + b \cdot \nabla_x u - |\sigma|^\alpha (-\Delta_x)^{\alpha/2} u + \ell = 0 \text{ on } Q_T, \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$

and

$$v_1(t,x) = \mathbb{E}^{t,x}\Big[\int_t^{\zeta \wedge T} \ell(s, X_s) ds\Big]$$

where $dX(t) = bdt + \sigma dJ_t$, and $\zeta = \tau_{\bar{O}}(X)$.

Cor Let O be a bounded open set satisfying exterior cone condition and $\sigma > 1$. If

$$\text{either } b = 0 \text{ or } \alpha \geq 1$$

Then the function v_1 is a viscosity solution of PDE.

Apply main result on
$w(t,x) = \mathbb{E}^y\Big[\int_0^{\zeta_1} e^{-\lambda r} \ell_1(Y_r) dr\Big]$,
$-\mathcal{L}_1 w(y) + \lambda w(y) - \ell_1(y) = 0$ on Q_T, and $w(y) = 0$ on $\mathcal{P}Q_T \cap \partial Q_T$, where
$Y_s = (t+s, X_{t+s}),\ \zeta_1 := \tau_{\bar{Q}_T}(Y),\ y = (t,x),\ \mathcal{L}_1 w(y) = (\partial_t u + \mathcal{L}_x u)(t,x),\ \ell_1(y) = e^{\lambda t}\ell(t,x)$

Nonlinear non-stationary equation

HJB equation

Consider solvability of, for $\gamma \geq 1$

$$-\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T, \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$

It is HJB equation, since

- If $\gamma = 1$, then write $-|\nabla_x u| = \inf_{b \in B_1}(b \cdot \nabla_x u)$,
- If $\gamma > 1$, then it is KPZ equation, also we write

$$-|\nabla_x u|^\gamma = \inf_{b \in \mathbb{R}^d}(-b \cdot \nabla u - L(b))$$

  with Lengendre transformation L of the function $F(x) = |x|^\gamma$

(Rmk) See control theory and HJB formulation to the references below

J. Ma and J. Yong, FBSDE and their Applications, 2007

B Oksendal and A. Sulem, Applied stochastic control of jump diffusions. 2007.

H. Pham, stochastic control with financial applications, 2009.

G. Yin and Q. Zhang Continuous-Time Markov Chains and Applications, 2013

J Yong and X Zhou. Stochastic controls, 1999

J Zhang. Backward stochastic differential equations, 2017.

Nonlinear non-stationary equation

Reducing solvability question by (CP + PM)

Consider solvability of, for $\gamma \geq 1$

$$-\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T, \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$

We assume that comparison principle and Perron's method hold:

- (CP+ PM) If there exists sub and supersolution, then PDE is uniquely solvable.

(Rmk) Existence of sub and supersolution may not be trivial, Exm 4.6 of [CIL92].

$Q_T := (0,T) \times O, \mathcal{P}Q_T := (0,T] \times \mathbb{R}^d \setminus Q_T$.

[CIL92] M Crandall, H Ishii, and P Lions. User's guide to viscosity solutions, *Bull. AMS*.

Nonlinear non-stationary equation

Existence of sub and supersolution

We want sub and supersolution of

$$-\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T, \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$

- $u = 0$ is supersolution.
- $v_1(t,x) = \mathbb{E}^{t,x}\left[\int_t^{\zeta \wedge T}(-1)ds\right]$ is a viscosity solution of

$$-\partial_t u + (-\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T, \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$

- Thus, v_1 is a subsolution of the nonlinear PDE.

Outline

Summary

- We prove that Feynman-Kac functional is a generalized viscosity solution of PDE under some conditions; It's the first attempt to connect generalized viscosity solution with fine topology.
- This can be applied to answer the existence of the viscosity solution of non-stationary Dirichlet problem;
- This idea can be extended to the solvability question by changing time into subordinate process.
- The discount rate can be removed if the integrability condition is added;
- Together with (CP + PM), this answers solvability questions for a class of nonlinear equations.
- Yet, we do not know if nonlinear Feynman-Kac functional is the Perron's solution.